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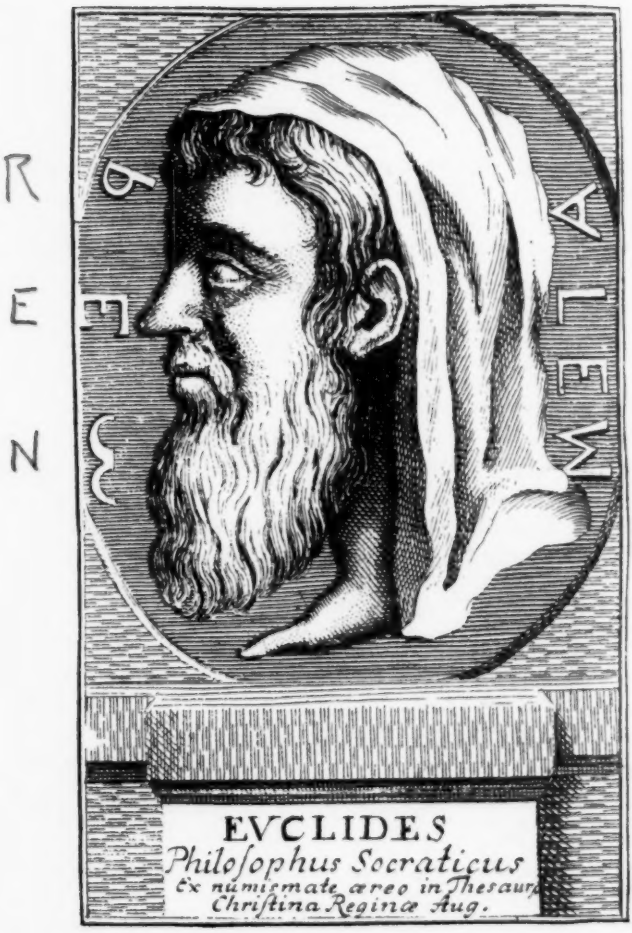

EUCLID—THE GREAT GEOMETER

Megarem.

= of Megara!

THE MATHEMATICS TEACHER

Volume XXIII No. 5

Edited by William David Reeve

Professors of Education and Their Academic Colleagues*

By WILLIAM C. BAGLEY

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I

AN OUTSTANDING feature of the present educational situation in the United States is the almost exclusive control of educational policies and programs in the lower schools by what I may call the "educational generalists." These comprise the professors of education in our universities, colleges, and teacher-training institutions, together with administrators, supervisors, and research-workers in the public-school service. The latter groups, in ever increasing numbers, represent in their attitude toward educational problems the traditions and ideals of the professors of education who are increasingly responsible for their training; hence the professor of education, in the last analysis, is the controlling agent. With the growth of his influence, the subject-matter specialists—the scholars in the differentiated fields of knowledge—have a constantly diminishing influence in determining where our educational system will go and how it will get there.

* A paper read at the Educational Conference of the Michigan State Normal College, Ypsilanti, Jan. 17, 1930; also before the National Society of College Teachers of Education, Atlantic City, Feb. 24, 1930.

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One can obtain convincing evidence of this statement by noting what has been happening during the past few years in the revision and reconstruction of elementary and high-school curricula. As we all know, our professional interests and enthusiasms have a habit of going in waves, and curriculum-reconstruction is not only the latest of these waves but also one that promises to influence the future development of American education far more profoundly than any of its predecessors. Throughout the country, committees have been and still are at work restating the objectives of education, reorganizing the materials, eliminating whole segments of subject-matter, selecting and organizing other segments to replace those that are abandoned, and integrating or trying to integrate materials from different subject-matter fields.

I am not questioning at this time the need for curriculum-readjustments of so fundamental and thoroughgoing a character. What I am emphasizing, however, is the almost complete absence of specialized scholarship in the deliberations out of which these changes are emerging—the almost complete absence of the subject-matter specialist from a council-table the potential significance of which to the future of our country it would be impossible to overestimate. The Curriculum Commission of the Department of Superintendence, which was in a fundamental way the clearing-house for the results of these deliberations, comprised thirteen members. Of these, two were professors of education, one was a teachers'-college president, and ten were public-school administrators, supervisors, and research-workers. There were no representatives of the specialized subject-matter fields on this commission. On the sub-committee coöperating with the commission in preparing the *Third Yearbook*,[1] there was not a single representative from the fields of history or geography; the sub-committee on reading had no representative from the field of literature; the committee on arithmetic had seven members, of whom five were educationists, psychologists, and research workers, while only two were specialists in the subject-matter field chiefly concerned.

The influential *Yearbooks*[2] of the National Society for the Study

[1] A very influential document of 422 pages, published by the National Education Association in 1925 under the title, "Research in Constructing the Elementary School Curriculum."

[2] The *Twenty-Sixth Yearbook*. (In two parts.) Bloomington, Illinois: The Public School Publishing Company, 1927.

of Education dealing with the curriculum problem were prepared under the general direction of a committee composed entirely of professors of education. The *Yearbooks* themselves comprised forty-four chapters written by thirty-two different contributors, not one of whom was a subject-matter scholar.

Professors of education and other educational generalists have, of course, a legitimate concern with the curricula of the lower schools. I doubt very seriously, however, whether they should exercise the almost exclusive control over elementary and secondary education that they exert to-day. This control, I am frank to say, is responsible in part for the looseness, not to say flabbiness, of our educational fibre. Under the proper conditions the influence of the subject-matter specialists, I am confident, could be a wholesome counter-agent to these weakening tendencies.

II

How can we account for the ever-widening influence of the professors of education in the public schools and the increasingly restricted influence of the subject-matter specialists? The answer to this question, I think, gives us an interesting perspective upon some of our present problems.

The most significant characteristic of American education is its essential democracy and universality. The remarkable and unprecedented growth in the enrollment of our high schools and the more recent expansion of the college and university enrollments together constitute a striking exemplification of the American educational ideal. Other countries have made elementary education universal; our country is very rapidly working toward universal secondary education and toward something that may ultimately be closely akin to universal higher education.

Quite obviously, this development has involved a quite new attitude toward both teaching problems and curricular problems. As long as the upper schools were essentially selective institutions, open only to the keener minds of each generation, diversified teaching skill was not of paramount importance. The relatively small and highly selected groups of pupils and students followed curricula made up of well-established and highly standardized materials. The teaching of these materials was also, on the whole, stabilized and standardized. Both materials and methods were adapted to the types of mind which

the higher school selected. If there was any lack of adaptation on the part of the learner, he was promptly eliminated. There was no thought of fitting either materials or methods to *his* needs and capacities.

The acceptance of the principle of universal secondary education, however, necessitated, as I have suggested, an entirely different attitude toward the teaching problem and toward curricular problems. Instead of eliminating the learner who could not master the standardized materials taught according to the long established methods, the effort was made to adapt both the materials and the methods to the learner's needs and capacities. New subjects were organized and incorporated in the curricula; the older subjects were in many ways modified; and the task of both the teacher of the older subjects and the teacher of the newer subjects was to find ways of presenting their materials that would make an appeal to the increasingly heterogeneous groups which crowded into the upper grades and the high schools.

It was largely for this reason that the need of a specialized training for high-school teachers came first clearly to be recognized. This was in the last two decades of the nineteenth century, just as the high schools were beginning to expand in a substantial way. Fifty years earlier, with the first real developments of universal elementary education, the need for a specialized preparation of teachers for the elementary schools had been recognized and met through the establishment of normal schools. When the ideal of universal secondary education generated a similar demand for the training of high-school teachers, the first answer came with the appointment of professors of education in the colleges and universities.

If this analysis of the situation is correct, it would seem that the professor of education, by historic tradition, represents the American ideal of mass-education as distinguished from the Old World ideal of selective or class-education. He owes his job to the need of adapting educational materials and teaching methods to the mass-mind—to the common mind of humanity. As long as the materials and methods of education were only those that had been adapted through a long process of evolution to a relatively small and highly selected group of learners, the professor of education would have been a luxury without a purpose. With the coming of the ideal of universal education, and particularly the ideal of universal secondary education, he became a very real necessity.

This interpretation explains several things. It explains, for example, why the United States, which has gone far beyond all other countries in extending mass-education upward, has more professors of education than all other countries combined. It goes a long way toward explaining why these professors of education exert to-day the powerful influence that they do in the work of the public schools. The general trend of the argument also offers an explanation for the recently developed but now rapidly expanding interest of college and university executives in teaching problems and curricular problems. Just as, a generation ago, the opening of the high school to the masses made necessary an entirely new attitude toward teaching and curricular problems on the secondary level, so the present expansion of the college and university enrollments is necessitating a new attitude toward these problems on the higher level. Only this year the first number of a new technical journal of higher education has appeared, with a professor of education[3] as its editor.

III

We come now to the question, Why has the subject-matter specialist played so slight a rôle in our recent developments on the elementary and secondary levels? If one were to reply to the question frankly and bluntly, the answer would be this: The subject-matter specialist has reflected and still tends to reflect the ideals of selective education rather than the ideals of mass-education. His primary interest has been in the development of his own specialized fields of knowledge rather than in the problem of making this knowledge part of the universal heritage. His interest in teaching, as such, has been a secondary interest, and he has not often generated very much enthusiasm over the task of stimulating the duller intellects. In practically every important college and university to-day, the status of the teacher is determined, not by his teaching, but by his contributions to the scholarly literature of his field.

I may put it in another way by saying that the arts college, like the graduate school, places the research-scholar on a much higher and more dignified pedestal than the teacher-scholar. The difference between the two types is a fundamental difference. The research-scholar is chiefly concerned with the vertical expansion of his specialized

[3] Professor W. W. Charters of Ohio State University. (This university publishes the new *Journal of Higher Education*.)

knowledge—in its growth upward through the discovery of new truths. The teacher-scholar is chiefly concerned with the horizontal expansion of his specialized knowledge—in its growth outward through making its truths in so far as possible a universal possession.

I am frank to say that, if there were more subject-matter specialists of the latter type—more teacher-scholars—the subject-matter fields would be much better and much more numerously represented at the council-tables where the policies and programs of American education are being formulated. That the professor of education now has a virtual monopoly of this strategic influence has been through no greed of his own. It is a monoply that has been almost literally handed over to him by his colleagues in the subject-matter fields. The latter have failed to grasp the tremendous significance of the transformation of the high school from an institution of selective class-education to an institution of non-selective mass-education. The professor of education, owing to his job as he does to this very transformation, has been in a position to sense its significance and to study seriously the far-reaching problems that it involves. As a result, his counsel has been in demand, his judgments have been respected, his advice has been followed.

IV

We come, however, to the very important fact that the job which has been delegated to the professor of education is far too complicated and far too important to be borne by him alone. It is a responsibility which should be shared by others and particularly by his colleagues in the subject-matter fields—if only they would be willing to interest themselves in the problems at issue! As long as they insist that there is no problem there, or as long as they insist that an advanced purely academic study of their subjects is all that they can offer to the solution of the problem, just so long will they be left out of the picture. They may be very sure that the lower schools will neither seek nor heed advice from people who nonchalantly regard the problems of teaching in the lower schools and organizing materials for the lower schools as unworthy of serious study.

Obviously, to give to subject-matter specialists the equipment that would enable them intelligently to grasp the educational problems involved in the lower schools would be a far simpler task than to

give to the educational generalist adequate subject-matter equipment in all of the specialized fields that he is called upon to deal with in developing teaching methods and curricular programs on the lower levels. From this point of view, the educationist's job can be at best but a bungling job; but so long as his academic colleagues hold themselves aloof, so long as they turn over to him the entire task of equipping prospective teachers with the attitudes, skills, and insights which will be needed in their professional work, just so long will the educationist continue to do the job as well as he can and just so long, even with the best of intentions on his part, will it be a bungling job.

V

As I have suggested, one trouble with the virtual control of the lower schools by the professor of education lies in his tendency toward a loosening of standards. This tendency has been natural enough. The vast upward expansion of universal education in our country would have been impossible without such a process. We could either relax standards and let the masses in, or retain standards and keep the masses out. Whether for better or for worse, we have taken the former alternative—and the masses, themselves, have played an important part in this decision. Personally, I believe that the decision will be for the better and not for the worse, but whether this is to be the case will depend upon whether we can make education on all of its levels an effective stimulus to mental growth for all who have the opportunity to benefit from it.

Now the professor of education, from the very nature of his job, has been obliged to justify the relaxation of standards. We find him very early in his career recommending the loose elective system as against rigorously prescribed curricula. We find him embracing the doctrine of interest and decrying the doctrine of effort. We find him discouraging promotional examinations and favoring admission to college on the basis of credits accumulated in the high school. We find him grasping with delight at the experimental results that seemed to invalidate the older doctrine of formal discipline, and generalizing these results far beyond any point justified by the experiments themselves. We find him about 1910 pointing to the newly discovered facts regarding retardation and elimination as proof that the standards which governed the progress of pupils through the schools were too rigid. (On the whole this was a salutary movement, although,

even at that time, the American elementary school was promoting to the secondary level a much larger proportion of its pupils than were the lower schools of other countries.) We find him in almost every case denying the educational value of practically every school subject that is difficult in the sense of being exact or exacting. We find him to-day leaning strongly toward the extreme freedom-theory of education which looks with suspicion upon the imposition of any learning task for which the learner does not have an immediate yearning and from which he cannot anticipate a pleasurable thrill.

In short, the professor of education, called upon to rationalize or justify a policy that would open the higher schools to the masses, has quite naturally overdone the job. He has carried the loosening process farther than the real needs of the situation demanded. Instead of recognizing a relaxation of standards as a necessary step in a fundamental process of readjustment, he has in effect attempted to justify loose standards *per se*. The practice of education in our country, I am convinced, may be made more virile, more rigorous, and more vigorous without in the least impeding or imperiling our educational expansion. The time has come, I am sure, for a corrective movement. There are signs within our own group of "educational generalists" that such a corrective movement[4] is being initiated; but in addition to what this will accomplish, education needs the balance and the stabilizing influence that the subject-matter specialists are in the best position to give if, as I have suggested, they (or at least some of them) will become thoroughly sensitive to the problem.

VI

I have been urging for a long time the adequate recognition of the teacher-scholar in the several fields of specialized subject-matter —the scholar who knows his subject in and out and all around, who loves his subject with a passionate devotion, who is keenly appreciative of what his field has contributed to civilization and of what it can contribute to mass-enlightenment, and who is interested above everything else in getting as many people as possible to know and love and appreciate his subject.

What has been called in cumbrous phrase the "professionalizing

[4] Professor H. C. Morrison's *Practice of Teaching in Secondary Schools* (University of Chicago Press, 1926), setting forth his "Mastery Formula," strikes a refreshingly virile note in educational theory.

of subject-matter" in the education of teachers is nothing more nor less than a plan to have the prospective teacher master the materials that will be his professional stock-in-trade under the leadership and inspiration of these teacher-scholars, and also gain an initial mastery of the teaching-art under the same guiding hand. At the present time, except in a few institutions, the prospective teacher is supposed to get his substantive stock-in-trade from "academic" instructors many of whom have but a slight acquaintance with the difficult problems of the lower schools and some of whom have neither sympathy for, nor appreciation of, the stupendous task that the lower schools are facing. At the same time, the instruction and training of the prospective teacher in the art of teaching is delegated to the educational generalist whose knowledge of the subject fields must necessarily be limited and superficial, and who is very likely wedded to a doctrinaire theory of education which impels him to cast all teaching problems under a single formula or set of formulæ.

What my proposal distinctly implies is that the fine art of teaching is not a mere bag of tricks from which the clever craftsman can conjure successive miracles. The tendency of general educational theory is to emphasize general patterns or type-procedures which, it assumes, may be fastened to any set of educational materials—or any type of subject-matter. At one time it is the "five normal steps" of the recitation; at another time, the "project-method"; at another time, the "contract" plan; and so on through the list. What my proposal emphasizes is the basic fact that the really effective techniques of teachings are closely related to the subject-matter with which they deal; that they grow out of a rich and vital understanding and appreciation of the subject to be taught, and have meaning and value only in terms of this subject-matter. The general patterns of teaching are not without their value, but it is, so to speak, a supplementary value to be realized best when the teacher is complete master of his substantive materials.

A prospective teacher's preparation should, in my judgment, center about a thoroughgoing study of the subject or subjects that he will teach and not about courses in the science and philosophy of education. The latter should be supplementary and complementary. His mastery of his materials should be as extensive and thoroughgoing as the time permits. I have long recommended for our normal schools and teachers' colleges a reduction in the time given to courses in

education and a much heavier emphasis upon the substantive courses, *under the proviso, however, that these be organized and taught with the needs of the prospective teacher constantly in view.*

This proviso simply means that such courses should be taught by teacher-scholars who are sensitive to the needs and problems of the lower schools. Give me one or two of these teacher-scholars in each of the major subject-matter fields and I shall not worry about the outcome.

On the other hand, the teacher-scholar if he is to take a central place in the professional education of the teacher, must be, no less than the professor of education, an educational democrat. It is the enlightenment of the masses that must be his passion. Let him be "high-hat," disdainful, or selectivistic, and, so far as a dynamic influence is concerned, his downfall is sealed. The professor of education still holds the trump-hand here, and, if his academic colleagues persist in blocking the game, he can still win. It might be a temporary victory for democracy. With the backing of his academic colleagues, with their insistence on discipline and system and order added to his insistence on mass-uplift, they can both win—and then there will be an enduring victory for democracy.

VII

In so far as I am aware, there is no single formula that will enable a subject-matter instructor to make his work of maximal helpfulness to prospective teachers. An emphasis of formulæ, indeed, would be more likely to hurt than to help the cause that I wish to promote. The essential element in the problem is idealistic rather than mechanistic. It depends primarily upon the recognition of teaching as fundamentally a fine art rather than a technological art.

At the same time, and with this reservation, I believe that the subject-matter instructor who wishes to qualify as a teacher-scholar will profit by training in the technology of education. Wherever careful experimentation has touched in a helpful way the problems of teaching in his particular field he should be familiar with the findings and competent to evaluate them in terms of the criteria that the scientific study of education has established. This implies, too, that he should be familiar with the applications of educational measurements to his field and with the principles and techniques of teaching that have a somewhat general application.

Even more important than his equipment on the side of educational technology, however, in his equipment in certain phases of his own field which the purely academic specialist often misses. I am confident that the teacher-scholar needs especially a knowledge and appreciation of the historical development of his subject. He needs this to the end that, in his own teaching and in that of his students later, the rich possibilities inherent in the genetic approach to the mastery of knowledge may be capitalized. In this connection, too, one should emphasize the importance of an appreciative understanding of the way in which one's specialized knowledge has influenced human welfare and human progress. The educational generalist has insisted that the knowledge which is made part of the educational pabulum must be humanized and vitalized and made to tingle with significance and meaning; and this is at least one demand of the educational generalist that the subject-matter specialist, if he wishes to become a real teacher-scholar, may well take to heart.

Above all else, I believe, the teacher-scholar must have a veritable passion for making his materials clear and stimulating, not only to the keener minds, but also to minds that are not quick to grasp new conceptions or to detect fine shades of meaning or to discern relationships readily. If he wishes inspiration toward the realization of this ideal I would refer him to the marked success of certain contemporary physicists and astronomers in bringing within the comprehension of non-mathematical minds the marvelous discoveries that have recently been made in their fields. When such eminent scientists as Arthur Stanley Eddington and Sir James Jeans can afford to devote their time and their talents to the horizontal expansion of their knowledge as well as to its vertical expansion, the teacher-scholar may safely conclude that the former function after all is of fundamental significance. And if the teacher-scholar wishes, not only inspiration but a splendid object-lesson in clear-cut exposition of the highest order and in the fine art of clarifying illustration, he can find it in the writings of the two men just referred to as it has been exemplified, I believe, in no other scientific writings since the time of John Tyndall.

VIII

What, finally, is the rôle of the "generalist" in educational progress? I have tried to show that his function is not to displace the subject-matter specialist; but this is only a restriction in the interest

of his own greater efficiency. The "educational generalist" is quite properly an integrating agent. He must see to it that, as far as may be, the various educational forces operate as a unit. The unity, however, cannot be forced, which is only another way of saying the generalist cannot, either individually or collectively, play the rôle of a dictator—and something akin to this rôle has been forced upon him by the circumstances that I have described. Educational policies are so pervasive and far-reaching that they should be determined only at council-tables about which are gathered representatives of every group that is concerned and of every group that can bring light to the deliberations.

And the educational "generalist" has a natural and proper sphere of activity in the problems that are common to all of the specialized fields. It is undoubtedly his function to study the larger aims that education should seek to realize, and to suggest the intermediate objectives the attainment of which, each in its turn, may help to realize these larger aims. But again he cannot be a dictator. The problem is too vast and too complicated—vaster and vastly more complicated than anything else in the world save possibly the problem of government itself. Over and over again the educational generalist has blundered because he has failed to take into account all of the factors and forces that affect his problem. Only a superman could envisage all of these factors and forces. A collective vision which reflected all the knowledge of all specialized groups would doubtless miss some of them. But the eye with a thousand lenses is much more likely to see a larger proportion of them than the eye with a single lens. The prime function of the educational generalist is to take the thousand pictures and try to blend them into a consistent whole. But this in itself is no child's play and the generalist will do well not only to get light from every possible source, including the subject-matter specialist, but to regard the completed picture as quite provisional in character, referring it back to its various sources for criticism and correction, so that, in the end, it will be the council-table as a whole that the picture reflects and represents.

A Professional Course for the Training of Geometry Teachers

By H. C. CHRISTOFFERSON
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IN THE TIME allotted for this paper it would be hopeless to attempt to give any comprehensive summary of a course for the training of geometry teachers. It will therefore be necessary to state very briefly and dogmatically certain features of that course, but in order that the presentation may not be criticized for being too sketchy to be useful, one portion at least will be given in more detail. The discussion naturally falls under three major headings, (1) objectives, (2) subject-matter, and (3) professionalization of subject-matter.

Before stating the objectives of the course for preparing teachers of geometry it will be necessary to state the general objectives of geometry itself. These will be stated briefly and with an attempt to give neither details nor defense.

Primarily geometry is taught, not merely for the information it imparts, but rather for the pattern of thought with which it provides experience. This pattern of thought or pattern of reasoning must give due regard for the function of definitions, postulates, logical sequence and proof. The informational portion of geometry is largely provided in the modern junior high school course. The simple materials and relationships of demonstrative geometry give it a peculiar function which is only secondarily informational. That inherent function is primarily and conspicuously to provide experience with drawing and proving conclusions concerning simple space relationships. Constant emphasis should be placed upon the relative nature of all proof and upon its dependence on definitions, postulates and previously proved conclusions.

The general objectives of any course for the training of teachers of high school geometry must then be (1) to insure familiarity with the materials and functions of high school geometry and (2) to establish a philosophy and certain principles of teaching which seem peculiarly dictated by the purpose of geometry and by the nature of its content.

With this brief statement of the general objectives of the course let us plunge immediately into a discussion of the subject-matter by means of which we hope to attain these objectives. That prospective teachers of geometry should have a thorough mastery of geometry is a postulate. It is also stated without proof that the college juniors or seniors preparing to teach high school mathematics ordinarily do not know their high school geometry well enough to teach it with confidence even though they have had two or three years of college mathematics.

Furthermore in most textbooks there are some 200 theorems, constructions and corollaries of plane geometry and about 150 theorems and corollaries of solid geometry, making considerably over 300 important principles, laws, or theorems in geometry, upon the basis of which original exercises are to be proved. Even in the National Committee's recommendation, counting converses, etc., there are over 160 propositions, to which the College Entrance Board adds 40 more, making over 200 as a minimum list of constructions, theorems and corollaries in geometry. To attempt to cover these in a professional course of half a semester, or approximately 27 hours, is impossible from the standpoint of real thoughtful mastery. It seems necessary, therefore, to select from these theorems a few for major emphasis and complete mastery. The selection must have due regard for sequence, for later usage and for importance. Based chiefly upon an analysis of later usage in geometry we have made such a selection. The analysis yielded a list of ten constructions, twenty theorems in plane geometry, and twelve theorems in solid geometry, which constitute the really fundamental or core constructions and theorems of geometry. These we have arranged in logical sequence, with the needed definitions and postulates. They constitute a perfectly logical course in geometry and really are the fundamental theorems upon which all of geometry can be built. Such a list might even have some possibilities in the high school to replace the needlessly long list of theorems now used.

It is not an impossible objective to require complete mastery by prospective geometry teachers of these ten simple constructions and thirty-two theorems of plane and solid geometry, and to place adequate emphasis upon the nature of proof, the function of postulates and the need for definitions. The following theorems with the necessary postulates, undefined terms and definitions constitute the course. The list of theorems make up the fundamental theorems of geometry. All

the remainder of plane and solid geometry depends directly upon these few theorems and can well be used as applications or original exercise material.

THE FUNDAMENTAL CONSTRUCTIONS AND THEOREMS OF GEOMETRY

I. Constructions of Plane Geometry

1. Construct a circle with a given radius and a given center.
2. Construct a triangle congruent to a given triangle using only the length of the three sides.
3. Construct an angle equal to a given angle.
4. Bisect a given angle.
5. Construct a perpendicular to a line at a point on the line.
6. Construct a triangle congruent to a given triangle using only two sides and their included angle.
7. Construct a triangle congruent to a given triangle using only one side and the two adjacent angles.
8. Construct a perpendicular bisector of a given line segment.
9. Construct a perpendicular to a line from a point not on the line.
10. Construct a line parallel to a given line through a given point.

II. Theorems on Straight Line Figures, Plane Geometry

1. The angles opposite the equal sides of an isosceles triangle are equal.
2. If two lines cut a third line so that the alternate interior angles are equal the lines are parallel.
3. The converse of 2. If two parallel lines cut a third line the alternate interior angles are equal.
4. The sum of the angles of any triangle is 180°.
5. The opposite sides and angles of a parallelogram are equal.
6. If three or more parallel lines cut off equal segments on one transversal they cut off equal segments on any transversal.
7. If a straight line is drawn through two sides of a triangle parallel to the third side it divides these sides proportionally.
8. Converse of 7. If a straight line divides two sides of a triangle. proportionally, it is parallel to the third side.

9. Two triangles ~~the~~ are similar if two angles of one are equal respectively to two angles of the other.
10. In any right triangle the perpendicular from the vertex of the right angle to the hypotenuse divides the triangle into two triangles, each similar to the given triangle.
11. In any right triangle the square on the hypotenuse is equal to the sum of the squares on the other two sides.
12. The area of a triangle is equal to half the ~~product of the~~ base times the altitude.
13. The locus of a point equally distant from two given points is the perpendicular bisector of the line segment joining the two points.
14. The locus of a point equally distant from two given intersecting lines is the pair of lines which bisect the angles formed by the given lines.

III. Theorems Concerning Circles, Plane Geometry

15. A diameter perpendicular to a chord bisects the chord and the arcs of the chord.
16. An angle inscribed in a circle is equal to half the central angle having the same arc.
17. A line perpendicular to a radius at its outer extremity is tangent to the circle at that point.
18. As the number of sides of a regular inscribed polygon is indefinitely increased its perimeter and area will both increase, while the perimeter and area of the circumscribed polygon formed by drawing tangents to the circle at the vertices of the inscribed polygon, will both decrease. The perimeters and areas of both polygons will each approach a limit.
19. The ratio between any circumference and its radius is constant and is equal to 2π.
20. The area of a circle is equal to π times the square of the radius.

IV. Fundamental Theorems of Solid Geometry

1. If two planes intersect, their intersection is a straight line.
2. If two parallel planes cut a third plane, the lines of intersection are parallel.
3. If two angles not in the same plane have their sides parallel in the same sense, the angles are equal.

4. The lateral area of a prism is the product of a lateral edge by the perimeter of a right section.
5. The volume of any prism equals the product of its base and altitude.
   a. An oblique prism is equal to a right prism whose base is a right section of the oblique prism and whose altitude is a lateral edge of the oblique prism.
   b. The volume of any parallelopiped equals the base times the altitude.
   c. A diagonal plane divides a parallelopiped into two equal triangular prisms.
   d. The volume of any triangular prism equals the product of its base and height.
   e. The volume of any prism is the product of its base and altitude.
6. The lateral area of a regular pyramid equals ½ the product of its slant height and the perimeter of the base.
7. If a pyramid is cut by a plane parallel to the base and a distance d from the vertex,
   a. The lateral edges and altitude h are divided proportionally.
   b. The section is similar to the base, the "ratio of similitude" being d/h.
   c. The ratio of the area of the section to the area of the base is d^2/h^2.
8. The volume of a pyramid equals ⅓ the product of its height and the area of its base.
   a. Two pyramids with equal bases and altitudes are equal.
   b. The volume of a triangular pyramid equals ⅓ bh.
   c. The volume of any pyramid equals ⅓ bh.
9. Lateral area of a cylinder equals the product of an element by the circumference of a right section, and the volume equals the product of its base and altitude.
10. Lateral area of a right circular cone equals ½ the product of the circumference of the base by the slant height and the volume of any cone equals ⅓ the product of the area of the base by the altitude.
11. The area of a sphere equals four π times the radius squared.
    a. The area of the frustrum of a cone of revolution equals ½ slant height times the circumference of a circle half way between the bases.

b. The area of a surface of revolution formed by revolving a regular polygon about a diameter is $2\ \pi$ times the apothem times the diameter.

c. The area of a sphere equals $4\ \pi$ times the radius squared.

12. The volume of a sphere equals $\frac{4}{3}\ \pi$ times the radius cubed.

Since with a college group two or three of these theorems can often be taken in a lesson, there exists then the possibility of using some time for emphasizing the pattern of thought that is so remarkably present in geometry. This would be impossible with a hurried covering of some 300 theorems.

Furthermore in order to furnish some additional background, several advanced theorems are also provided, including inequalities and incommensurables, the theorems of Ceva and Menelaus, Euler's line and the Nine-point circle, Coaxal circles, and reference is given to projective geometry, the geometry of inversion and non-Euclidian geometry.

From the standpoint of subject-matter the mastery of the content of the course just outlined insures full mastery of the materials of high school geometry plus. It now remains to show how professionalization is effected. Before doing this it seems necessary to emphasize another feature of subject-matter.

Several times in this paper mention has been made of the crucial place of definitions in reasoning. If evolutionists and non-evolutionists could agree on a definition of evolution and confine their arguments to the issue thus defined, much of the literature on the subject would be abandoned as irrelevant. Bryan's evolution is not the scientist's evolution. Lack of clear definition is the cause of much argument that is futile. Even geometry teachers and geometry textbooks sometimes fail to distinguish definitions and proof. Many texts prove so-called theorems which are really definitions. For instance the circumference or length of a circle must really be defined as the limit of the perimeter of its inscribed polygon as the number of its sides is indefinitely increased. Units of length are straight line segments; therefore, the length of a curve can mean nothing except the limit of a series of straight line segments. The only thing that can be proved about the circumference is that the above described limit exists and that the limit is expressible in a formula which makes its computation possible.

Similar errors are made in the proofs of areas and volumes of all

curved figures of plane and solid geometry. Need for careful definition is also apparent in the case of parallels, parallelograms, similar figures, tangents, incommensurables and a host of other terms. The distinction between definition, postulate and proof is often difficult; however, the distinction is of the utmost importance not only for reasoning in geometry but also for reasoning in science and in life. Teachers need to be trained to use the utmost discrimination in dealing with these fine distinctions.

Finally the subject-matter is professionalized. The professionalization is based upon three postulates of education. Postulate 1. Most teachers teach very largely as they have been taught rather than as they have been told to teach. Postulate 2. We learn to do by doing. Postulate 3. It is as possible to make a silken purse from a sow's ear as to teach a subject that one does not know.

In realization of the truth of postulate 1, namely, that teachers teach largely as they are taught, a general pattern of teaching geometry is illustrated in the course. For want of a better name we will call it heuristic teaching, although it is more widely heuristic than most plans so labelled. The word heuristic comes from a Greek word meaning "I find." The subject-matter of geometry is so peculiarly adapted to heuristic teaching that it fairly dictates such a procedure. In general the heuristic plan we use includes induction, deduction, analysis, and synthesis. It may be used with individual, group, or laboratory work in geometry. The plan involves the following steps of procedure. By a study of specific cases we have the students discover a general conclusion worth proving. That is by induction we discover deduction. Then, by analysis of this conclusion they discover the relationships upon which the proof depends. Finally, they arrange those relationships in synthetic deductive form to constitute a concise proof. One or two illustrations will help to make the technique clear.

Given a proportion such as $\frac{2}{3}=\frac{10}{15}$ it is remarkable what you can and cannot do with it by changing the order of the numbers and adding some of them together or subtracting, etc. If $\frac{2}{3}$ equals $\frac{10}{15}$ then

1. $\dfrac{2}{10}=\dfrac{3}{15}$

2. $\dfrac{2+3}{3}=\dfrac{10+15}{15}$

3. $\dfrac{2+10}{10}=\dfrac{3+15}{15}$

4. $\dfrac{2+10}{2}=\dfrac{3+15}{3}$

5. $\frac{2-10}{10}=\frac{3-15}{15}$ 8. $\frac{2-3}{2}=\frac{10-15}{10}$

6. $\frac{2-10}{2}=\frac{3-15}{3}$ 9. $2\times 15=3\times 10$

7. $\frac{2-3}{3}=\frac{10-15}{15}$ 10. $\frac{2+10}{3+15}=\frac{2}{3}$ etc.

These relations are all true, as can easily be shown by simplifying. Would the corresponding relations be true of any proportion? How would you state relation 3 above so that it would apply to any proportion? Answer: "In any proportion the sum of the first and third terms over the third equals the sum of the second and fourth over the fourth." Or

"if (1) $\frac{a}{b}=\frac{c}{d}$ then (2) $\frac{a+c}{c}=\frac{b+d}{d}$."

If this statement can be proved then relation (3) will be true of any proportion. Let us then analyze the conclusion to discover upon what it depends.

(2) $\frac{a+c}{c}=\frac{b+d}{d}$ if (3) $(a+c)d=(b+d)c$.

(3) is true if (4) $ad+cd=bc+cd$.

(4) is true if (5) $ad=bc$. But $ad=bc$ from (1) by clearing of fractions. Therefore we can start with (1) clear fractions to get (5), add cd to get (4), factor to get (3), divide by cd to get (2). Or synthetically—

(1) $\frac{a}{b}=\frac{c}{d}$ given.

(2) $ad=bc$ multiplying (1) by bd

(3) $ad+cd=bc+cd$ adding cd to (2)

(4) $(a+c)d=(b+d)c$ factoring (3)

Therefore (5) $\frac{a+c}{c}=\frac{b+d}{d}$ dividing (4) by cd

Therefore if $\frac{a}{b}=\frac{c}{d}$ then $\frac{a+c}{c}=\frac{b+d}{d}$.

Note the induction, deduction, analysis, and synthesis and also note how naturally they follow each other, each supplementing the other.

For the second illustration let us take something more geometric and not so familiar. Again notice the inductive approach leading to the discovery of the deductive conclusion. Then the analysis of the conclusion resulting in the discovery of the proof which is finally stated in synthetic form.

Approach: Draw several triangles and locate O, G, and H, that is, the intersection of the perpendicular bisectors, of the medians, and of the altitudes.

Do you notice any relationship which seems to be true of these points in all these triangles? Are they collinear? Do OG and OH have any definite ratio? State the general conclusion. Answer: "In any triangle O, G, and H are collinear and HG equals ⅔ of HO."

Analysis: G seems to divide HO in the ratio of 2 to 1 just as it does the medians. Therefore if the three points are collinear then the conclusion suggests the use of similar triangles having HG and GO as corresponding sides and the segments of a median as another pair

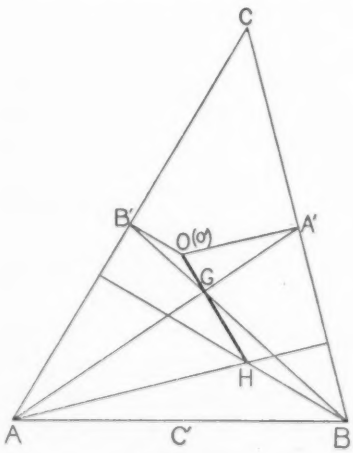

Figure 1

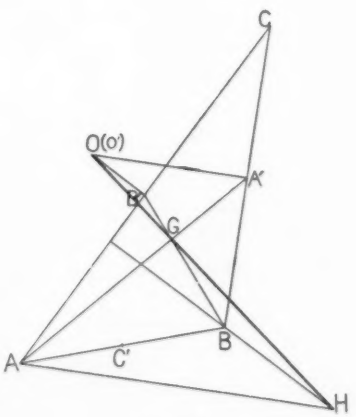

Figure 2

of corresponding sides. To prove the points collinear extend HG and prove O is on it or extend OG and prove H is on it, or even draw OH and prove G is on it. Finally if HG is extended to O′ making HG = ⅔ HO′, and then if O′ could be proved to be O the proof would be easily completed. O′ could be shown to be O by showing that O′A′ and O′B′ are the perpendicular bisectors of BC and AC.

Proof: (1) Draw AA′, AH, BB′, and BH and extend HG to O′ making GO′ = ½ HG. Draw lines O′A′ and O′B′.

(2) Then triangles A′O′G and AHG are similar. Why?

(3) Therefore angles A′O′G and AHG are equal. Why?

(4) Therefore A′O′ is perpendicular to BC at A′ being parallel to AH which is perpendicular to BC.

(5) Similarly B′O′ is perpendicular to AC at B′.

(6) Therefore O′ is O, and O, G and H are collinear and so related that HG = ⅔ HO.

As a third illustration of this heuristic technique I shall report a lesson taught by one of our practice teachers. She started with this direction: (1) "Draw a line, on it lay off at least three equal segments and through the points of division draw a series of parallel lines." Notice that this is a laboratory inductive approach. (2) "What is true of these parallel lines?" Answers: (a) "They are the same distance apart." (b) "If you drew a second line across them the segments on it would be equal."

(3) "Will someone state the last conclusion in a sentence which includes the hypothesis. This sentence will be a new theorem which will be very useful." While this was being acceptably done and recorded a figure like figure 3 appeared on the board and then these questions were asked.

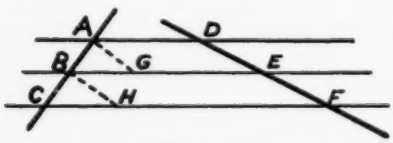

Figure 3

(4) "What is the hypothesis?" Answered.

(5) "What is the conclusion?" Answered.

(6) "We seem to need to prove two line segments equal. What methods do we have for doing this?" Answers: "Congruent triangles" and "parallelograms."

(7) "We seem to have neither of these conditions here yet. Do you have a suggestion then on how to prove DE = EF?"

Two hands went up at once and several others in quick succession. The teacher called on one and the following is almost an exact quotation, "If you draw lines through A and B parallel to DF the two triangles formed will be congruent. Then DE would equal EF because they are opposite sides of parallelograms whose other sides are equal."

This is really the whole proof. It was more clearly stated by more detailed questions and finally the students were directed to write it up in synthetic form as a proof. Someone suggested drawing the parallels on the other side in the usual way. That was accepted as possible. Time does not permit a complete description of the lesson, but an attempt has been made to show the heuristic features. First, the heuristic approach resulting in the discovery of a conclusion worth proving. Second, the analysis resulting in a most remarkable discovery of the proof in a new form which really surprised the teacher. Third, the natural interplay of induction, deduction, analysis and synthesis in a presentation partly laboratory and partly group discussion. This illustrates the functioning of the first postulates of professionalization of subject-matter, that teachers teach as they are taught.

The other postulates of professionalization have already been used in requiring a mastery of the subject-matter of high school geometry as well as some advanced geometry in order that prospective teachers may learn to do by doing and also in order that they may teach with a confidence resulting from complete familiarity with the subject.

In this brief presentation an effort has been made to show (1) the objectives of a professional course in geometry, (2) the subject-matter used and its means of selection, (3) the pattern of teaching use to professionalize that subject-matter so as to stimulate high ideals of teaching and so as to establish certain heuristic principles or techniques.

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The Value of Examination Papers to the Teacher

By JOSEPH B. ORLEANS

Chairman, Mathematics Department, George Washington High School, New York City

(Continued from the March number.)

THE ARTICLE in the March number of the Mathematics Teacher contained a summary of the correct and the incorrect answers to questions on tests in algebra and of typical errors found in the papers written by the pupils. This paper contains similar tabulations for geometry.

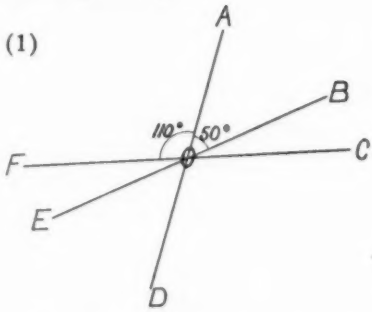
Teachers of mathematics, I think, generally agree that geometry is more difficult to teach than algebra. In the latter, no matter how much the teacher rationalizes a lesson in its presentation and in its development, the work sooner or later becomes mechanical. With the proper amount of drill the pupil very soon learns what reaction to make to the algebraic expression that is presented to him and he conceals the rationalization in such expressions as "transpose," "find the L.C.D.," and "clear of fractions." And even many of the algebraic problems have, thanks to some of the textbook writers, been reduced to types that can be done mechanically by filling in the carefully prepared rectangles in a definite way.

Not so in geometry. Here, even in the simplest exercise, the pupil must analyze the problem with a series of questions by means of which he rejects and selects his material and works out a method of procedure. This thought process is, therefore, a slow and difficult one for many of the pupils who fill up the classes in our high schools. It is a very common thing to see pupils groping in the dark for a period of from five to ten weeks before they find themselves and realize what the work is about.

A summary of the errors made by pupils in geometry ought to be quite interesting for one who cares to study the reaction of the pupil to the work as it is presented throughout the term and most valuable to the teacher who seeks to know what to improve and how to improve it in a second presentation of each topic.

The following is a list of the number of correct and incorrect replies to a set of simple questions at the end of a few weeks of instruction in geometry.

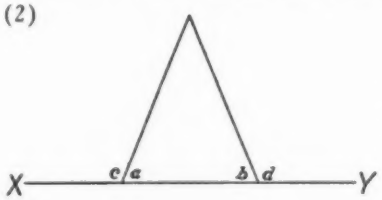
(1)



	Right	*Wrong*
Angle BOC contains .....°	96	108
Angle EOC contains .....°	112	92
Angle AOE contains .....°	87	117
Angle EOD contains .....°	125	79
Angle FOD contains .....°	85	119

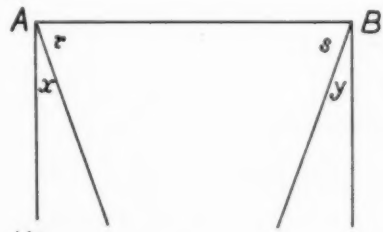
Why is finding the size of angles BOC, AOE and FOD more difficult than angles EOC, and EOD?

(2)



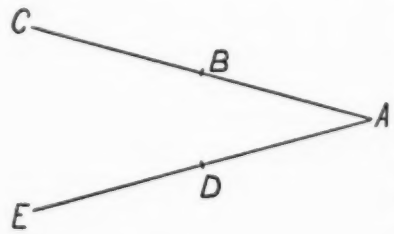
If angle a = angle b and XY is a straight line, then angle c = angle d because...	110	94

(3)



If angle A = 90° and angle B = 90° and angle x = angle y then angle r = angle s because	125	79

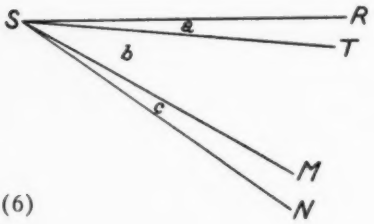
(4)



If AC = AE and B is the midpoint of AC and D is the midpoint of AE, then AB = AD because	95	109
and AB = BC because	93	111

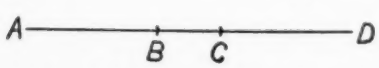
Why was it so difficult for the pupils to differentiate between "bisection" and "halves of equals"?

(5)



	Right	*Wrong*
If angle a = angle c, then angle RSM = angle NST because.	100	104

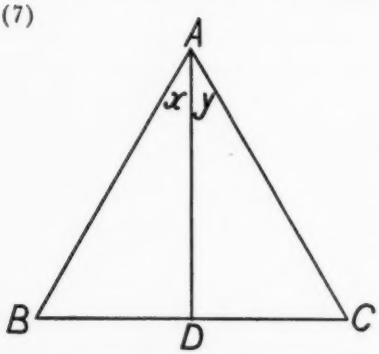
(6)



If AB = CD, then AC = BD because.	138	66

Why did the pupils find no. 5 more difficult than no. 6?

(7)

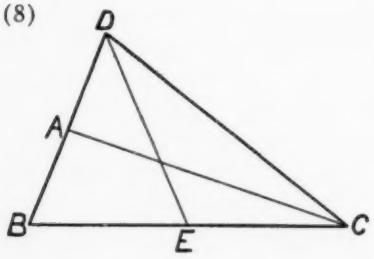


Given: AB = AC and AD bisects angle BAC. *Prove:* Δ BAD ≅ Δ ADC	76	128

(The proof of this exercise was called wrong if it contained any omission or error at all.)

Why did the pupils find it difficult to prove correctly such a simple exercise?

(8)



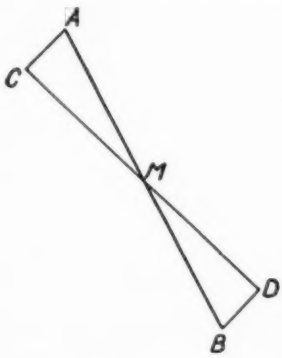
If AC bisects BD, then line .. = line	220	51
If angle BDC is bisected, then angle .. = angle	121	150

What is there more difficult about understanding the bisection of an angle than about the bisection of a line?

		Right	*Wrong*
(9)	The complement of 50 degrees is	201	70
	To find the supplement of an angle you must	103	168

If a teacher stops to analyze a summary like this one and is able to formulate answers to the questions, she cannot help bringing about an improvement in her work.

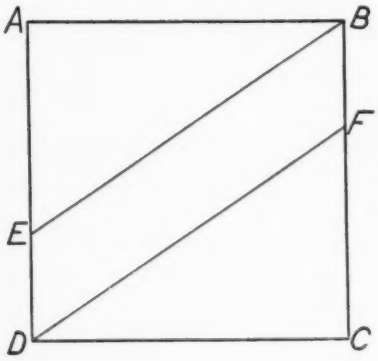
The following is a list of types of errors made by pupils in answer to certain questions on geometry tests. The figures in parentheses indicate the number of pupils who made the error.



		Right	*Wrong*
	Lines AB and CD are bisected at M. Angles C and D are right angles. Therefore,		
	(a) AC is perpendicular to .. and .. is perpendicular to CD.....	365	215
	(b) AM = .. and CM = ...	551	29
	(c) One pair of vertical angles is .. and the other pair is	320	260

Answers:

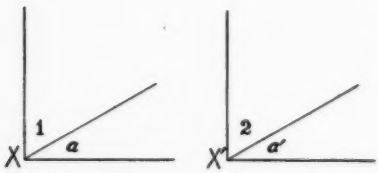
(a)	AC ⊥ AB and AB ⊥ CD.	(73)	(c)	Omitted	(68)
	AC ⊥ AB	(71)		B and A	(55)
	AC ⊥ CD	(27)		ACM and BDM	(33)
	AB ⊥ CD	(9)		AMB and CMD	(12)
	AC ⊥ DB	(5)		A and C	(9)
	AC ⊥ M and B ⊥ CD	(4)		DBM	(9)
				AB and CD	(10)
	Omitted	(17)		AC and DB	(7)
(b)	CM = CD	(8)		AM and MB	(7)
	CM = MB	(4)		MDB	(6)
	AM = MC	(2)		ACM and DBM	(5)
	AM = MD	(4)		CAM and BMA	(4)
	Omitted	(9)		Correct pair of angles repeated	(20)



	Right	*Wrong*
In the square ABCD, BF = DE. Therefore FC = AE because	321	259

Answers:

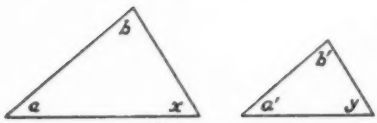
Corresponding parts of congruent triangles	(76)
Addition	(65)
Omitted	(34)
Halves of equals are equal	(26)
Sides of a square are equal	(17)
Right angles are equal	(12)
Equal parts of equals are equal	(6)
Things equal to the same thing are equal to each other	(4)
Supplements of equal angles are equal	(3)



Angle a = angle a'. Angles X and X' are right angles. Therefore what is true about angles 1 and 2?	457	123

Answers:

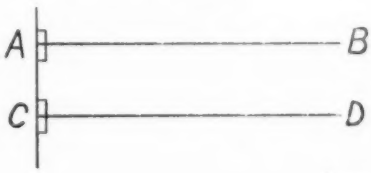
Angles 1 and 2 are right angles	(20)
Complementary angles	(23)
1 is a complement of a and 2 of a'	(14)
Supplements of equal angles are equal	(17)
Omitted	(12)
Corresponding parts of congruent triangles	(5)
1 and 2 are acute	(4)
1 and 2 are supplementary	(3)



	Right	*Wrong*
If angle a = a′ and angle b = b′, then x = y because.....	190	81

Answers:

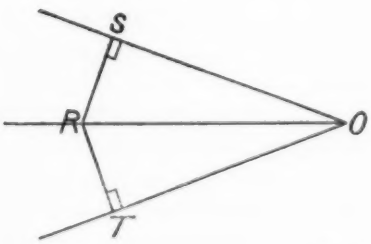
Omitted	(20)
Corresponding parts of congruent triangles	(13)
Base angles of an isosceles triangle are equal	(10)
Base angles of congruent triangles are equal	(3)
Things equal to the same or equal things etc.	(6)
All angles of an equiangular triangle are equal	(2)
Complements of the same or equal angles are equal	(2)
No matter how long the sides of an angle are extended they will equal the same	(1)



AB is parallel to CD because	224	47

Answers:

If two parallel lines are cut by a transversal a pair of equal corresponding angles are formed	(5)
If alternate interior angles are equal, the lines are parallel	(3)
When two lines have their right angles on a straight line, they are equal	(2)

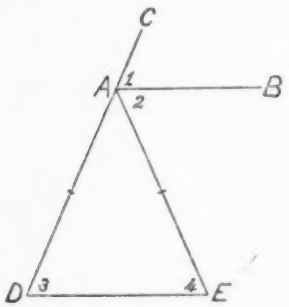


If RS = RT, then RO	181	90

Answers:

Is the perpendicular bisector	(15)
Identity	(15)

Bisects ST (19)
Bisects SRT (4)
SO = OT (2)
Is the hypotenuse (6)
Omitted (13)



AB bisects angle CAE
triangle ADE is isosceles.
Fill in items 1, 2, 3 and 4.

	Right	*Wrong*
	104	167

1.....
2.....
3.....
4.....
5 Therefore AB is parallel to DF.

Answers:

Omitted angle CAE = ∠ 3 + ∠ 4 (128)
Omitted angle 1 = angle 2 (2)
Stated only angle 1 = angle 3 (2)
Stated only angle 3 = angle 4 (2)

Such a list can be made indefinitely long and the uses to which it may be put are many. It ought to prove interesting and very much worth while for a teacher to try to understand why certain errors were made and to analyze the thought process that might have led to them. Such an analysis will not only give the teacher a basis on which to plan her work but also make her feel more certain that she is dealing with some children who cannot possibly handle the subject matter that is being given to them and will furnish her with material that should convince the supervisor that the only solution of the problem in the teaching of geometry is the scientific grading of the classes and the use of modified courses of study fitted to the abilities of the pupils.

How Much Do Pupils Enjoy Mathematics?

By CHARLES H. BUTLER
University High School, University of Missouri, Columbia, Mo.

SOME TIME AGO the writer endeavored to get some idea of the extent to which students in one Junior-Senior High School *enjoyed* work in mathematics. The inquiry took the form of a questionnaire in which the students were asked to answer eight specific questions.

The Questionnaire. The questions were framed in such a way as to reduce the possibility of ambiguity to a minimum, and as a further safeguard against misinterpretation of answers each student was specifically requested to answer each question by either "yes" or "no." As a further guard against the possibility of students giving answers which they might think were desired instead of those that represented their situations and attitudes honestly, they were informed that this was not a test but merely a means of collecting information. They were told that since their replies to the questions would in no way affect their standings in their school work, there would be no advantage in giving answers that were not strictly truthful.

The results seem to indicate that for the most part, at least, the students answered the questions in this spirit, a considerable percentage openly declaring that things mathematical held no interest for them. The results are of course subject to the general limitations of the questionnaire method, and in particular to those of the questionnaire which does not call for exact quantitative answers based on records. It is believed, however, that this limitation is in large part offset by the fact that apparently all the pupils took the matter seriously and answered the questions honestly according to their best judgment. Pending evidence to the contrary, we may for the present assume that the results of the survey represent accurately the existing situation with respect to mathematical enjoyment among the pupils of this school. The questionnaire is shown below.

(About 10 minutes) INFORMATION BLANK (not a test) MATHEMATICAL ENJOYMENT

Name .....
.....
Date .....
.....

This is not really a test, but merely some questions for you to answer out of your own experience. Be perfectly frank and truthful, because these tests have nothing to do with your class work nor your grades. Just answer each question as carefully and as accurately as you can. Start on the first question right now.

1. Have you done any mathematical work or reading or thinking any time in the last two weeks *just because you liked to do it?* (Answer yes or no) .....

If your answer to *Question 1* was "no" do not write anything below this line. If your answer was "yes," then answer the following questions as exactly as you can.

2. Do you often notice and think about the geometrical figures that occur in buildings, automobiles, streets, and other things that you see every day? .....

3. Do you like to read articles that contain graphs, tables of statistics, and that talk about numbers and quantities? .....

4. Do you ever work problems *just for fun?* .....

5. Do you ever "think up" mathematical problems just for fun? .....

6. Have you really enjoyed the mathematics courses which you have taken in school? .....

7. Do you want to take some more mathematics in school? .....

8. Do you think mathematics is harder than the other school subjects? .....

The Subjects. Seventy-three pupils answered the questionnaire. The sex distribution was almost even: thirty-five boys and thirty-eight girls. The grade distribution of this group is shown in the following table:

Grade	7	8	9	10	11	12	Total
Boys	3	6	13	8	2	3	35
Girls	6	3	8	12	2	7	38
Total	9	9	21	20	4	10	73

While this does not include more than about three-fourths of the total student body of the school, it represents pretty well a prorated class distribution.

The Results. Question I was answered by every one of the seventy-three subjects. This is especially fortunate because it is the most comprehensive and generally illuminating of all the questions. A careful study of the answers to this question was therefore made,

and separate tabulations were made for the Senior High School and Junior High School groups. This was not done for the other seven questions for two reasons: (1) because they were, in a sense, subordinate questions, and (2) because the replies were incomplete.

Question 1 was worded as follows: "Have you done any mathematical work or reading or thinking any time in the last two weeks *just because you liked* to do it?" (Answer Yes or No)———— The tabulation of the answers to this question is shown below:

	Yes	No	% Yes
Junior High School	27	11	71%
Senior High School	21	14	60%
Total	48	25	65%

There are two rather surprising things about this table. The first is that so large a percentage of the whole student body is sufficiently interested in anything having any connection with mathematics to even think about it voluntarily. If the replies are honest, the situation should indeed be encouraging to the mathematics teacher.

The second surprising thing is that the Junior High School group should show a larger percentage of affirmative replies than the Senior High School group. Normally we expect the latter to be a more highly selected group of people than will be found in the Junior High School, and mathematical interest will usually run about in proportion to the selectivity of the group. As a matter of fact, the latter is the case here, the unusual element being that in this instance the Junior High School group is more highly selected than the Senior High School group.

Summary. In summarizing the results a tabulation by sex was made for the purpose of discovering whether there was any pronounced difference between the sex groups with reference to mathematical enjoyment. A slight difference between the groups as wholes was found to be in favor of the boys, 62% of whose replies were "yes" as against 60% for the girls. This difference, however, was regarded as being too small to have any material significance.

The answers of the whole group of students to all of the questions are summarized in the following table:

Question	1	2	3	4	5	6	7	8	Total
Yes	48	35	23	40	22	42	43	26	279
No	25	15	30	15	32	12	10	31	170

Not Answering	0	23	20	18	19	19	20	16	135
Total	73	73	73	73	73	73	73	73	73
Per cent "Yes"	66	48	31	55	30	58	59	36	48

Thus, in Question 3, 23 answers were affirmative, 30 were negative, and 20 pupils failed to answer the question. Hence 31% of all the students notice geometrical forms in architecture, and 69% evidently do not. The other columns in the table are to be read in the same fashion.

Taking the whole series of questions and the whole group in summary, we find that so far as our questionnaire covers the field for mathematical enjoyment, and so far as our group is representative of the whole school, about 48% of the possibilities for mathematical enjoyment and appreciation are realized, at least in some measure. This is a higher percentage than the investigator had expected to find, but it still leaves a vast field for constructive and useful work in this direction on the part of the mathematics teachers and curriculum makers in the secondary schools.

The writer is of the opinion (born largely of experience and observation) that the fate of the thing called mathematical appreciation depends more largely upon the teacher than upon any other single factor or combination of factors. If this is true, the question as to whether it shall survive and grow and flourish will find its answer in the enthusiasm, skill, ingenuity, insight, and sympathy of the teacher rather than in anything else.

Thither Algebra

By P. STROUP
West High School, Cleveland, Ohio

THE RECENT ARTICLE by Mr. Betz is a classic of timeliness and wisdom. It arouses the faintly breathing hope that something may yet be done. Mark Twain's remark about the weather might well be applied to the situation in mathematics. There is still however an observation which is perhaps worth making. Our splendid physical prosperity must have some educational basis which is partially responsible for it. Now surely if there is a weakness in our present national development it is spiritual rather than material, it is in culture rather than in mechanical advances. So we are compelled to ask those who raise the cry that we have not taught mathematics efficiently enough just what they have a right to expect. They have set up certain expectations for various stages of mathematical development in the child and have been disappointed to test and find their expectations are not met. Perhaps they were optimistic rather than wise. The technical schools may claim all the credit but it still leaves nothing to complain about as far as the mechanical side of our celebrated prosperity is concerned. The thing that we need most to be concerned about is whether those that succeeded in mathematics furnished a smaller per cent of bootleggers and divorcees than those who failed. I have no statistics to offer but I have always pursued my work as a teacher of elementary algebra and geometry encouraged by the hope that if my students were led to think straighter in mathematics they would have a greater chance of leading useful happy lives. No one has come right out and said that the faulty teaching of algebra and the rest of the debatable high school subjects were responsible for the increase in crime except perhaps Henry Ford with his announcement of millions for education. His proposals do not sound feasible enough as a general solution to warrant our sitting back and waiting for his specifications for the reforming of our high schools.

In response to a paper a few years ago by Miss Marie Gugle on the introduction of more college mathematics into the high school

curriculum I predicted that that wave of development had about reached its climax and that the counter wave of popular high school education would stop it and perhaps drive it back. Mr. Breslich when asked recently how much calculus was being taught in high schools expressed the opinion that though there was considerable talk about it in "papers" little was being done about it in curricula. Eighth grade algebra has had its day. Ninth grade algebra is retreating. It took in too much territory and spread itself so thin that now it is forced to retreat on every front. The ninth grade is not the place for the algebra that used to be beginning algebra to be given as a required subject. It is no longer required of everyone and the algebra itself is diluted. The demand for dilution has not been laid with sufficient candor at the door of changed high school population. Referring to the beginning algebra of 30 years ago as undiluted I think it is safe to assert that it was fairly well taught. At least those who studied it then were competent to carry on and advance our present prosperity and though the number of failures was large it was not alarmingly so as it later became. The curtailments in algebra such as the nests of parentheses and artificial radicals I do not mourn, and yet those who were really ready to begin algebra were not stopped by those excrescences. The introduction of commercial graphs beyond the amount needed to introduce straight line graphs was pure dilution and it was welcomed because it was something "teachable" rather than because of the conviction that beginning algebra was the place for it. Economics or a separate course would be a more appropriate place for the extended treatment of graphs given in some algebras. Surely the content of algebra that needed teaching was sufficient without a short study of statistical presentation. The ease with which the graph was assimilated by the pupils of limited intelligence merely indicates that commercial graphs are as graphical as the name indicates they ought to be.

The question I want seriously to raise is whether this diluted algebra is worth the time spent on it from any point of view. As the matter now stands in my school the third semester of algebra which used to be college preparatory is now the algebra of the first semester of the eleventh grade and college preparatory algebra comes in the twelfth grade. This means that the pupils who used to be prepared for college in three semesters are wasting a semester of their time; for surely we get some pupils today that are as bright

as the majority of those who used to take algebra. There is no report from the colleges that the students are better prepared than formerly. The teaching of ninth grade algebra is not what it used to be in its effects on the brighter students and it is not the teachers who are at fault. The content has been diluted to reduce the number of failures and the standards have been lowered. The third semester of algebra has to be taught much more slowly and in a much more elementary manner than previously. We are still required to cover the same number of subjects that we did but they are touched on so lightly it is small wonder that when a general review examination is given the students are so confused that the reflection on the teaching is very uncomfortable to say the least. The diluted algebra can have only three lines of defense, vocational, cultural, and endured because nothing better has been presented. This "drilled in" algebra can claim little as a preparation for future mathematicians and can claim nothing as a general preparation for happy useful living. It maintains its present position by reason of educational conservatism and lack of competition. I cannot agree with Mr. Betz that the perfection of teaching in beginning algebra in the ninth grade to all comers no matter how perfect it becomes can ever solve the problem. Classification has failed through inherent weaknesses or impracticability of administration. The public has quietly squelched whatever relief differentiated schools might have offered by insisting on a college certificate for every graduate. What remains? First, beginning algebra must be undiluted and taught in the way that Mr. Betz has designated. But when and to whom? My answer would be "to those who are ready for it when they are ready." We have turned from teaching it in a way that profited some to a way that profits none. I think that the teaching of algebra thirty years ago was better than at the present time and the descent was forced by the endeavor to reduce the number of failures. It can be as truly said that there is no plebian road to mathematics as that there is no royal road. The idea that by sufficiently graduated drills minds not capable of understanding algebra can be trained into an understanding of it has caused tiresome algebras to be written and much conscientious teaching effort to be wasted. The cause is only a little less hopeless than the proposal to train chickens to live on sawdust by gradual substitution. From the standpoint of evolution I suppose that is possible but it is a matter of generations not of individuals.

I propose that beginning algebra be taught only for the benefit of those ready to assimilate it and not stamped as ninth grade mathematics. Those ninth graders who would take it would be selected by fine performance in a general intelligence test or in their previous work. If the parents insist or the child is ambitious though not gifted let him try it perhaps as an extra subject and if success seems probable the alternate subject could be dropped or vice versa. When the child has become a tenth grader he could try it again or try it for the first time. Putting off the algebra pending further development or change of attitude is worth trying. I have a feeling as I work with students in third semester algebra that I could really teach them algebra if there was time to practice more. When they did their practice in beginning algebra the good effects were lost because they did not understand. The drill I refer to here is essential to making habits of processes thoroughly understood. Their habits of non-understanding imitation lead them constantly into difficulties. The objection that it is too early in the life of the student to limit his possibilities is outweighed by the consideration that the beginning of algebra before the student is ready for it is a waste of time or worse because it generates the wrong attitude toward mathematics and reduces the probability of ultimate successful accomplishment. Thus there would be eleventh and twelfth grade students taking beginning algebra. Then or never is certainly the time for many of our present students to begin algebra. Withdrawals would solve a large part of this problem even as they do now.

What are the students to do while they are not taking algebra? Subjects must be developed which have factual content of value and also furnish materials for thinking. The facts of algebra are worthless if not understood. The facts of biology, botany, and geology while furnishing food for the deepest thought can be presented in an interesting and valuable way without being as thoroughly understood as the facts of algebra must be. Let them be presented from the point of view of furnishing avocations. As we go further into this machine age that point of view must receive more emphasis. Mr. Eastman is wise indeed in insisting on the importance of educating our people in the proper enjoyment of leisure time of which whether they are successful in amassing wealth or not they are bound to have more and more as we progress in the substitution of machine for manual labor. Mr. Eastman's espousal of music for this purpose

suggests that we can use music for the same purpose. I am not greatly embarrassed by the laughs that greet this suggestion for this cannot be denied. The manual practice needed in instrumental performances can be acquired only by practice and the best time for this practice is just this time in the life of the child while the drill in algebra unless accompanied by adequate understanding is a complete loss if not a detriment. As we progress into the machine age with the promise of more leisure why hurry the process of education as if the industrial world were reaching ravenously for our product. Why not take a little more time and do a better job?

More could also be made of the study of the principles and practical care of the automobile by both sexes not only by those who may use such knowledge as a vocation but also by those who may use it by choice in spare time or by necessity at sundry times. This suggests also a course in the actual care and repair of household machinery. These subjects furnish facts which have value more or less superficially memorized but also furnish food for thought for those who have the stuff to think with. Elementary civics and economics could be developed with a study of local institutions. A dull boy taken on a sightseeing tour through the water works would get more for his time than on a sight seeing tour through algebra. The students that quit before graduation would have more for the time spent than under an algebra-Latin curriculum which trains them in the habit of failure and the attitude of not expecting to understand. If some students acquire enough credits to graduate but they are not of the right kind for the next step in their education, they can take the required work in an additional semester if possible by concentrated effort. That would surely be better than an extra semester occasioned by failures.

On the whole the proposition of the public schools educating for leisure and good citizenship rather than for the job has more to recommend it than the converse. In the apprenticeship the routine memorization that is necessary is sure not to be neglected but if education and apprenticeship are largely preparation for the job the successful use of leisure is left to chance. If every child that passes through the high school should emerge with a hobby or the nucleus of one to profitably or at least safely occupy his spare time the schools would justify their existence on a dollar basis. I do not propose making over the entire curriculum on this basis, nor do I hold that our

present students are starved of this element but I am trying to justify it as a worthy objective among others if possible and all by itself if necessary. In athletics where it would seem to fit most naturally it governs the situation almost none at all. Gardening and art should unblushingly stress this objective. Cooking for boys should not be as rare as it is.

The attitude and demands of the public will have to be modified. That can be done. Mr. Betz has shown the way. The newspapers if sold on the matter can revolutionize or at least cause a temporary revolution in public thinking. In making these suggestions I am acutely aware of the impossibility of seeing the future. We will not get out of the bog unless we try some road. If we are led into another bog it may be a better one than this. Wise heads and true hearts led us into this one but here we are. What shall we do?

Additional Names for the Register of Officers and Members

The following names and addresses were either omitted or incorrectly given in the recent Register of Officers and Members of the National Council of Teachers of Mathematics:

Batt, A. Laura
19 Church St.
West Concord, Mass.

Dice, Elizabeth
North Dallas High School
Dallas, Texas

Harrell, Professor J. W.
Baylor University
Waco, Texas

Kranz, Gertrude E.
282 Aycrigg Ave.
Passaic, N.J.

Miller, A. Brown
3295 Avalon Rd.
Cleveland, Ohio.

Slatter, Florence
Box 120
Downey, Calif.

Worden, Orpha
Detroit Teachers College
Detroit, Mich.

Wyatt, Ruth
3523 Sunnyside Ave.
Philadelphia, Pa.

An Algebra Baseball Game*

THE PLAY INSTINCT is strong in the child—work to him is drudgery. Have you ever noticed how hard a child works, perhaps all day long, raking the leaves with which he builds a leaf-house, or tugging logs to build an Indian fort? Or, in winter, does he mind the cold when he is building a snowman, or when he is skating or ice-boating? That is play, but when the boy's mother asks him to rake the leaves or shovel the snow, that is quite a different question—that is work.

It is the duty of every teacher to so vitalize the work that school is life to her pupils so that work becomes a pleasure rather than drudgery. It is my purpose to discuss one phase in vitalizing algebra so as to make the study of beginning algebra seem a privilege and a pleasure, not a requirement and a burden, that is, by introducing the play element in the form of games. Interest and enthusiasm are always created when there is a little friendly rivalry, and there is always a certain pride in having one's side win. In teaching such topics as "Signed Numbers," I have used games or contests that proved very successful, but the best one I have ever used was in the review of factoring—an Algebra Baseball Game, which is played as nearly in accordance with baseball rules as is practical.

RULES

1. The class room constitutes the "diamond."
2. The "baseballs" are problems to be factored.
3. The "umpire" is the teacher or anyone else who is qualified for this important position.
4. Two "captains" are chosen by the class, and each captain picks his "line-up"—"pitcher," "catcher," "short-stop," "basemen," "fielders," and "substitutes." Each captain manages his own team and the team which develops the best "team work" is certain to win.
5. A "score keeper" is appointed by each captain, and his duty is to record all runs, outs, etc.; and take care of the "score board."
6. Each captain also appoints a "time keeper" whose duties will be later defined.

* Some of our readers would like to hear of more games like this one that really work, especially for use in Mathematics Clubs. We shall be glad to receive any others. Whoever sent this article in forgot to sign his name. We shall be glad to give credit to the author if he will tell us who he is.—EDITOR

7 The number of "innings" played is at the discretion of the teacher, and this depends upon whether the game moves rapidly, whether interest and enthusiasm run high, and whether the results accomplished warrant a nine-inning game.

8. The "balls" (problems) are chosen by the two teams, each "pitcher" having the list of problems for his team, and "pitching" from that list. When the "umpire" calls, "Play ball," both teams must be in position to play and the pitcher is allowed only one minute in which to pitch his first ball (problem). He must read the problem clearly and accurately, and while he reads the problem, the "batter" copies it on the blackboard. If the problem is read inaccurately or not clearly, the umpire calls it a "ball," and the pitcher will proceed with his second problem. The umpire shall call a "ball" on the pitcher each time he delays the game by failing to deliver the ball to the batsman for a longer period than twenty seconds, excepting at the beginning of an inning (when he may have one minute). The umpire shall also call a "ball" on the pitcher if he gives the same problem twice to a batter in any one inning.

9. As in baseball, a "strike" is a "pitched ball struck at by the batsman," that is, any problem that is attempted by the batter whether or not it is completed, or "a fair ball delivered by the pitcher at which the batsman does not strike." The batter may refuse both the first and second balls but if he refuses the third, the catcher takes the problem. If the catcher "catches" it (solves it correctly in the prescribed time), the batter is "out," if not, the batter goes to first base.

10. A "foul ball" is a problem that is incorrect or partially solved in the allotted prescribed time. When a batter makes a "foul," the pitcher designates the catcher or anyone else on his team to catch it. If the ball is "caught" (solved correctly in the prescribed time) the batter is out. If not, the batter is given another ball. After "three outs" made by the team "at bat," the team in the field takes the place of the one at bat and *vice versa*.

11. A problem that requires only one line of factoring constitutes a "first base" problem; one with two lines, a "second base" problem, and so on. (If a problem requires more than four lines of factoring, it shall be called a "ball" by the umpire.) Then too, the time element is important and must be considered. For a "first base" problem, fifteen seconds are allowed for solving after the problem is written on

the blackboard; thirty seconds are allowed for a "second base"; forty-five seconds for a "third," and one minute for a "home run."

12. Each player of the side "at bat" shall become the batsman in order in which his name appears in the "batting list." Before the game, the batting order of each team must be handed in by the captain to the umpire, and the umpire shall submit it to the captain of the other team. This order must be followed throughout the game, unless a player be substituted, in which case the substitute must take the place in the batting order of the retired player. After the first inning, the first batter in each inning shall be the one whose name follows that of the last batter in the preceding inning.

13. "A substitute may at any stage of the game take the place of a player whose name is in the team's batting order." The captain may also at any time ask players on his team to exchange positions if he considers such a change desirable. Whenever one player is substituted for another, the captain who makes the change must immediately notify the umpire, who in turn must announce the same to the spectators.

14. The batter becomes a "base-runner" immediately after he makes a "fair hit," when "four balls" have been called by the umpire, or when the catcher has failed to catch the ball on the third strike. The base runner is entitled to advance a base when another batsman becomes a base runner and forces him off the base. (There can be only one runner on each base.)

15. The pitcher may at any time attempt to put a runner out on base except when the umpire calls, "Play ball," when he must deliver the next ball to the batter. (The umpire calls "Play ball" after the pitcher has attempted to put out one runner, whether or not he succeeds.) To put out a base runner, the pitcher uses a one-line problem. He calls the runner by name, and the runner steps to the board and copies the problem as dictated. If he solves it correctly in the specified time (15 seconds), he is called "safe" by the umpire. If the runner fails to solve it, the problem is then given to the baseman. If the baseman solves it correctly, the runner is "out," that is, "caught off base"; if the baseman fails, the runner is called "safe." In case the pitcher gives a two-line problem by mistake, and the runner solves it, he advances a base. If the runner fails, the pitcher may call on either baseman to "catch it"; if the baseman fails, the runner is called "safe" on the base of the one designated to catch it.

16. In case a runner attempts to "steal a base," the pitcher directs his next ball to that runner, and the same procedure is used as that prescribed in the preceding paragraph.

17. A "run" is scored every time a base-runner returns to the home base before three players are put out.

18. "Under no circumstances shall a captain or player dispute the accuracy of the umpire's judgment and decision on a play."

The following is a specific example:

Suppose Batter X "pitches" the following problem to Batter I: Factor $a^4 - 256$.

This requires two lines of factoring:

$$a^4 - 256 = (a^2 + 16)(a^2 - 16)$$
$$= (a^2 + 16)(a + 4)(a - 4)$$

If it is solved correctly and on time, Batter I goes to second base, and Batter II takes his turn. Suppose the following problem is given: Factor $3x^2 + 30x + 27$. If Batter II solves it, he goes to second base, and Batter I is forced to go to third. If Batter II had solved a three-line problem, Batter II would go to third base, and Batter I would score a "run."

Suppose Batter III knows very little about factoring. If he is not sure of solving the first problem given him, he refuses it, and this is called "Strike One." The second may also appeal to him as being difficult, so he is wise to refuse it also. By this time the pitcher may have grown tired and may give him an easy one. If the third is another difficult one, the batter has to think quickly to decide which is the more advantageous—to attempt it or not. If he refuses it, the catcher takes it. If the catcher solves it, Batter III is "out"; if he fails Batter III goes to first base. If Batter III attempts it and fails, the pitcher may designate anyone to take it.

Geometric Proofs for Trigonometric Formulas

By ARTHUR HAAS

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Brooklyn, New York

THE FOLLOWING rather interesting method of proving geometrically the trigonometric formulas for the sum and difference of two angles, which involves a very simple construction, seems not to have claimed the attention of text book writers in mathematics. Furthermore, no one to whom I have shown it, seems to recall having seen it anywhere else, so I submit herewith to the readers of *The Mathematics Teacher*.

The advantages of my proofs lie in the simplicity of the diagrams and in the fact that they are more general than usual. (No special diagram and proof needed for the second quadrant.) The disadvantage is that it is a "proof" rather than a development.

I should like to add that one of our high school pupils, (Aaron Hirschfeld) succeeded in working out the proofs for the tangents independently, after having had the diagram suggested to him.

THEOREM I

The sine of the sum of two angles is equal to the sum of the product of the sine of the first by the cosine of the second and the product of the sine of the second by the cosine of the first.

This means that we must prove that

$$\sin(x + y) = \sin x \cos y + \cos x \sin y.$$

Construction: Construct $\triangle ABC$ (Plate 1) with angles x and y, (whose sum is less than 180°) as two of its angles. Prolong AB and draw CH perpendicular to AB prolonged. Draw altitude BK to AC. Then

1. In $\triangle BHC$ $\sin(x+y)=h/a$ while
2. In $\triangle ABK$ $\sin x=l/b$ and
3. $\cos x=c/b$.

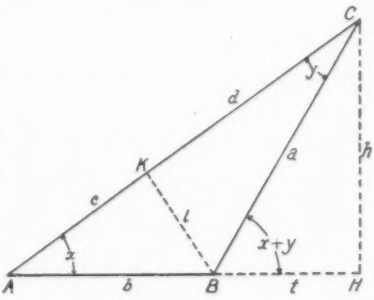


Plate 1
For Theorems I, II, & III

Similarly

4. In $\triangle KCB$ $\sin y = l/a$ and

5. $\cos y = d/a$.

Therefore, $$\sin x \cos y + \cos x \sin y = \frac{ld}{ab} + \frac{lc}{ab} = \frac{l(c+d)}{ab}$$

But $l(c+d)$ = twice the area of the $\triangle ABC$ $= hb$ and therefore

$$\sin x \cos y + \cos x \sin y = \frac{hb}{ab} = \frac{h}{a}$$

and therefore $\sin (x+y) = \sin x \cos y + \cos x \sin y$.

THEOREM II

The cosine of the sum of two angles equals the difference between the product of the cosines and product of the sines of those angles.

Here we must prove that $\cos (x+y) = \cos x \cos y - \sin x \sin y$.

The Construction and Diagram are the same as for Theorem I.

1. In $\triangle BCH$ $\cos (x+y) = t/a$.

Equations 2, 3, 4, 5 as in Theorem 1 should follow here.

Therefore,

6. $$\cos x \cos y - \sin x \sin y = \frac{cd}{ba} - \frac{l^2}{ba} = \frac{cd - l^2}{ab}$$

But in $\triangle ABK$, and AHC, $\angle x$ is common, and $\angle AKB = \angle H$. Therefore, the triangles are similar and the following proportion is true:

$$\frac{c+d}{b+t} = \frac{b}{c}$$

Then $c^2 + cd = b^2 + bt$ and $c^2 - b^2 + cd = bt$, but $b^2 - c^2; = l^2$; therefore $cd - l^2 = bt$. Finally, substituting in Equation 6 above;

$$\cos x \cos y - \sin x \sin y = \frac{bt}{ab} = \frac{t}{a}$$

whence, substituting in (1)

$$\cos (x+y) = \cos x \cos y - \sin x \sin y.$$

THEOREM III

The tangent of the sum of two angles equals the sum of their tangents, divided by the difference between unity and the product of their tangents.

Here we must prove that:

$$\tan (x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}.$$

The construction and diagram are the same as for Theorem I.

1. In $\triangle BCH$, $\tan (x+y) = h/t$,
2. In $\triangle AKB$, $\tan x = l/c$
3. And in $\triangle BCK$, $\tan y = l/d$ then

$$\frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{l/c + l/d}{1 - l^2/cd}$$

$$= \frac{\dfrac{l(c + d)}{cd}}{\dfrac{cd - l^2}{cd}}$$

$$= \frac{l(c + d)}{cd - l^2}$$

But $l(c+d) = hb$ (see Theorem I) and $cd - l^2 = bt$ (see Theorem II) Therefore

$$\frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{hb}{tb} = \frac{h}{t}$$

therefore

$$\tan (x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

THEOREM IV

The sine of the difference of two angles equals the difference of the product of the sine of the first by the cosine of the second and the cosine of the first by the sine of the second.

Let x and y be the two angles ($\angle x < 180°$; and $\angle y < \angle x$). We must then prove that

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

Construction: At some point, B, in line AH draw BC; making $\angle CBH = \angle x$; at another point, A, draw AC; making $\angle CAB = \angle y$;

and let these two lines meet at C. Construct $CH \perp$ to AB prolonged, and draw $BK \perp$ to AC.

$$\angle C = \angle x - \angle y.$$

Proof:

1. In $\triangle CBK$, $\sin(x-y) = l/a$
2. Also in $\triangle BCH$ $\sin x = h/a$

and

3. $\cos x = t/a$

while

4. In $\triangle ABK$ $\sin y = l/b$

and

5. $\cos y = c/b$

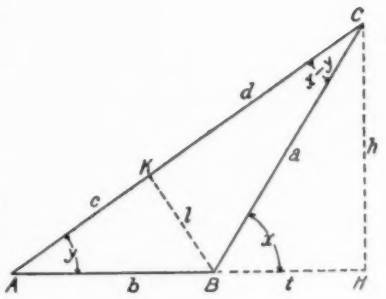


Plate 2
For Theorems IV, V, & VI

6. Therefore $\sin x \cos y - \cos x \sin y = \dfrac{hc}{ab} - \dfrac{lt}{ab} = \dfrac{hc - lt}{ab}$

but in $\triangle ABK$ and ACH, $\angle y$ is common and K and H are right angles. Therefore $\triangle ABK \sim \triangle ACH$ and therefore

$$\frac{c}{b+t} = \frac{l}{h}$$

whence

$$ch = bl + tl$$

or

$$ch - tl = bl$$

and

$$\frac{hc - lt}{ab} = \frac{bl}{ab} = \frac{l}{a}$$

therefore, substituting in Equation 6

$$\sin(x - y) = \sin x \cos y - \cos x \sin y$$

THEOREM V

The cosine of the difference of two angles is equal to the sum of the product of the cosines and the product of the sines of those angles.

Here we must prove that

$$\cos (x - y) = \cos x \cos y + \sin x \sin y$$

The construction and diagram are the same as for Theorem IV.
Proof:

1. $\cos (x - y) = d/a$
2. $\cos x = t/a$
3. $\cos y = \dfrac{b + t}{c + d}$
4. $\sin x = h/a$
5. $\sin y = \dfrac{h}{c + d}$

therefore

6. $\cos x \cos y + \sin x \sin y = \dfrac{t(b + t)}{a(c + d)} + \dfrac{h^2}{a(c + d)} = \dfrac{bt + t^2 + h^2}{a(c + d)}$

but

7. $(b^2 + 2bt + t^2) + h^2 = c^2 + 2cd + d^2$

by the Pythagorean theorem.

Also

8. $\dfrac{b}{c + d} = \dfrac{c}{b + t}$

Homologous sides of similar triangles.

Whence;

9. $b^2 + bt = c^2 + cd$

Subtracting (9) from (7) we have

10. $bt + t^2 + h^2 = cd + d^2$

Substituting this value in Equation 6

$$\cos x \cos y + \sin x \sin y = \frac{cd + d^2}{a(c + d)} = \frac{d(c + d)}{a(c + d)} = \frac{d}{a}$$

therefore, substituting in (1) we have

$$\cos (x - y) = \cos x \cos y + \sin x \sin y$$

THEOREM VI

The tangent of the difference between two angles equals the difference between the tangents of those angles divided by the sum of their products and unity.

Here we must prove that

$$\tan x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

The construction and diagram are the same as for Theorem IV

1. In $\triangle CBK$ $\tan (x-y)=l/d$

also

2. In $\triangle CBH$ $\tan x=h/t$

and

3. In $\triangle ABK$ $\tan y=h/b+t$ therefore

$$\frac{\tan x - \tan y}{1 + \tan x \tan y} = \frac{\frac{h}{t} - \frac{h}{b+t}}{1+\frac{bh}{t(b+t)}}$$

$$= \frac{\frac{bh + th - th}{t(b+t)}}{\frac{bt + t^2 + bh}{t(b+t)}}$$

$$= \frac{bh}{bt + t^2 + bh}$$

but $bh=l(c+d)$ (see Theorem I) and $bt+t^2+bh=cd+d^2$ (see Theorem V, Equation 9) therefore

$$\frac{\tan x - \tan y}{1 + \tan x \tan y} = \frac{l(c+d)}{d(c+d)} = \frac{l}{d}$$

therefore

$$\tan (x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

Report of the Retiring President

By H. C. Barber
Exeter, N.H.

Organization

The National Council makes possible the organization of all our teachers of secondary mathematics into a unit. Such organization is important. It increases the opportunity for formulation and study of our professional problems. It helps us to take our place among educational organizations in this country and abroad. It enables us to speak with considerable effectiveness in regard to educational matters which concern us. It makes possible the continuance of the campaign initiated by The National Committee on Mathematical Requirements.

Accomplishments

The National Council is progressing. Its membership grows rapidly. More and more local clubs and associations are holding out welcoming hands, affiliating with the Council, giving and receiving help. It will be found, I think, that all cities which are habitually represented at our annual meetings are improving their courses of study in mathematics.

The administrative machinery of the Council functions smoothly with the labor and responsibility divided among the President and Board of Directors, the permanent Secretary-Treasurer, and the Committee on the Official Journal. The Mathematics Teacher and the Yearbooks are gaining readers and influence.

The Council is recognized by and affiliated with the Mathematical Association of America. Members of the Council who meet the Association's requirements for admission may be admitted without payment of the initiation fee. The Council and the Association have appointed a joint committee which is studying the advisability of reorganizing the present courses in geometry.

Looking Ahead

We ought to increase our membership to 10,000 at least, not for the sake of mere size but so that we can make still greater improve-

ment in our journal, so that we can help more teachers, and so that we can increase our influence in behalf of better teaching.

We ought to become more and more a clearing house for ideas; stimulating discussion on the part of local associations, bringing the discussion to a head, acting upon its conclusions.

We ought to help raise the standard of admission to our profession. We ought to wage a battle against routine teaching and in behalf of courses and methods which are intellectually stimulating. We ought to convince teachers and public that mathematics is essential in education for modern life.

Teaching

I had rather earn my living by teaching than in any other way. In my mind, teaching is not merely a life work, a profession, an occupation, a struggle; it is a passion. I love to teach. I love to teach as a painter loves to paint, as a musician loves to play, as a singer loves to sing, as a strong man loves to run a race. Teaching is an art—an art so great and so difficult to master that a man or woman can spend a long life at it, without realizing much more than his limitations and his mistakes, and his distance from the ideal. But the main aim of my happy days has been to become a good teacher, just as every good architect wishes to be a good architect, and every professional poet strives toward perfection.

—William Lyon Phelps

The New President

of the National Council of Teachers of Mathematics.

A Message from the New President of the National Council

By JOHN P. EVERETT
Western State Teachers College, Kalamazoo, Mich.

JUST ONE century ago the colleges of America were in process of recognizing and accepting algebra and geometry as appropriate subjects to be studied and presented by candidates for higher education as evidence of their fitness for admission to institutions of higher education. Just a decade ago there was organized the National Council of Teachers of Mathematics. None of us took part in the earlier movement: very few of us were present in Cleveland when the Council held its first meeting in 1920: yet both events very properly engage our interest.

When secondary mathematics was first placed on the list of preparatory subjects at Harvard University it shared its place with three other subjects: Greek, Latin and geography. Within the interval of the past one hundred years the original restricted list of preparatory subjects has grown to include more than a score of well differentiated topics. The main ideal of secondary instructors has changed from preparation for college to *education* of all children. The development of mechanical power has transformed our social and economic relationships from an agrarian to an industrial civilization.

We all know what significant changes have been wrought in the comparative importance of Greek, Latin, and geography in our secondary schools, but the place of mathematics in the curriculum has been affected the least of any subject. Neither the threat of obsolesence nor the competition of modern subjects have served to keep mathematics from having been, and being now, considered as of prime importance. In 1925 an investigation made under the auspices of the North Central Association of Colleges and Secondary schools, covering nineteen states and including 678,935 pupils, showed that first-year English was the subject studied by the greatest number, but algebra stood next with 975 enrolled for every 1,000 enrolled in first-year English. All this constitutes a remarkable record

of recognition by society and of service to society. Well may teachers of mathematics feel a thrill of pride in the position which their subject has long held in education. A backward look is both inspiring and necessary if we are to have a proper sense of the orientation of our work, but after all, the National Council of Teachers of Mathematics is essentially a forward looking organization. Mathematics, however glorious its history, cannot expect long to hold its place unless it can continue to contribute both to material prosperity and to spiritual growth of today and tomorrow.

The fundamental laws of mathematics have not changed since the days of Euclid, but new laws have been discovered, and teaching must proceed in the light of, and with regard for, the scientific knowledge of the present moment. These considerations alone render probable, and frequently imperative, a constant readjustment of values and methods. Human nature changes little, or not at all, but human nature whether in the family or the community, whether in the workshop or the field of recreation, is now called upon to meet more situations that are new within a generation than a millenium formerly envisaged. No static scheme of education suffices for such a time as ours.

The National Council exists not to perpetuate a régime, but to afford a medium for the expression and exercise of the highest type of cooperative endeavor in an attempt to discover what the best interests of youth and adulthood demand and what are the best methods of serving those interests. The programs of the annual meeting, THE MATHEMATICS TEACHER, and the Year Books are genuine contributions to the solution of educational problems that confront us all right now.

Industrialism demands not less, but constantly more and more, mathematical knowledge and ability. Just now there is a pronounced disposition to emphasize the practicability of mathematics. This is very well unless we allow the corollary to gain credence that only *practical* mathematics should be taught. Those educators who insist that children should be taught only useful things overlook the fact that human progress since the dawn of civilization has been due not to emphasis upon what has been essential at any given time, but to the encouragement that the mind has had to use its imagination and to invade not only the realm of the impractical, but even of the impossible.

An educational philosophy that fixes the content of the curriculum according to usage would have kept the Hindu-Arabic system of notation from Western civilization to this very day. No teacher can afford not to be enthusiastic over his own subject, but also no teacher can afford to be narrow in his outlook. With understanding of subject matter there needs to go a progressive appreciation of the place of that subject matter in the entire scheme of things. It is in both of these fields that such an organization as the National Council of Teachers of Mathematics makes a constructive contribution.

A membership of over 5,600 is a striking indication of professional alertness on the part of teachers of mathematics. When a group met in a small room in Cleveland ten short years ago and initiated the movement that has developed into the National Council, I doubt if anyone present on that occasion could have foreseen the enthusiastic cooperation which the idea of a professional, voluntary association of teachers of mathematics would receive.

Ten thousand members is one of our immediate objectives. That number does not look at all extravagant in view of what has already been accomplished. We should work for an increased membership. We should also strive to retain a sense of individual responsibility, even in the presence of great size. Ask any teacher of mathematics, after taking a look at the list of topics in the "Classified Index" of sixty-eight pages that has just been issued, if membership in a group that can make such a contribution is not worth both the small membership fee and loyal moral support as well.

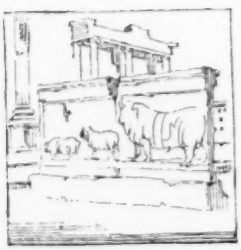
You and I and all of us may well be happy to think that teachers of mathematics have been, are, and will continue to be, ready to do their share in making education the most vital force which intelligent and consecrated effort can produce.

Euclid the Great Geometer

MANY GREAT names have been associated with Alexandria but of these that of Euclid is the greatest. "He was the most successful textbook writer that the world has ever known, over one thousand editions of his geometry having appeared in print since 1482, and manuscripts of this work having dominated the teaching of the subject for eighteen hundred years preceding that time. He is the only man to whom there ever came or ever can come again the glory of having successfully incorporated in his own writings all the essential parts of the accumulated mathematical knowledge of his time."

We know little of Euclid's early life. He may have studied in the schools founded by the great philosophers Plato and Aristotle at Athens, in Greece. He became head of the mathematics school at Alexandria, Egypt, and proceeded to collect and organize into a set form the known geometric principles. He is said to have insisted on the knowledge of geometry for its own sake. Thus, we read of his telling the youthful Prince Ptolemy, "There is no royal road to geometry." At another time, so the story goes, when a lad who had just begun geometry asked, "What do I gain by learning all this stuff?" Euclid made his slave girl give the boy some coppers, "since," said he, "he must make a profit out of what he learns."

Geometry took definite form as a science when Euclid (about 300 B.C.) wrote his "Elements of Geometry." The proofs of his text were so excellent that the book replaced all other texts of the time and has held an influential position to this day. The form of Euclid is practically the same as most American geometry texts, and in England boys still say they are studying Euclid (meaning geometry).



NEW BOOKS

Training in High School Mathematics Essential for Success in Certain College Subjects (With Especial Reference to Physics). By Allan Ray Congdon. Teachers College Contributions to Education No. 403. Bureau of Publications, Teachers College, Columbia University, New York. 1930, pp. 110, $1.75.

The purpose of this study, as stated by the author, "is to determine what training in high school mathematics is essential for certain college subjects that do not presuppose a knowledge of college mathematics. 'Training in high school mathematics' is interpreted to include the various outcomes from the study of algebra, geometry, and trigonometry, such as facts, concepts, skills, general processes, and methods of procedure."

The procedure used by the author in realizing this purpose is best stated probably in his own words. "The text entitled, *Physics, A Textbook for Colleges,* written by Oscar M. Stewart is used as a basis of study in making the following investigations:

1. The mathematical vocabulary used has been determined by a careful word count in the entire text.

2. The symbolism employed in the text has been tabulated, and compared with that used in four other texts in freshman physics, and five texts in freshman chemistry.

3. By means of an intensive study of the entire text, including the complete solution of the five hundred seventy-two problems presented in the text, the facts, concepts, skills, general processes, and methods of procedure which are used have been analyzed and tabulated.

4. From the list of algebraic manipulations, eighteen problems have been selected to constitute a test. The results of giving this test to eight hundred fifty-nine high school pupils who had studied algebra for at least three semesters, have been tabulated and discussed".

This procedure has been carried through rather carefully and it is quite evident from a reading of the book that much careful and painstaking work has been done. Many of us, who have been taking certain things for granted in reference to the way the things we have taught in secondary mathematics are applied to college subjects, especially physics, are much surprised to discover that Professor Congdon failed to find them at all in the books he studied, or if found, they were not at all in the forms we have been teaching. It will mean, if we accept his conclusions, that we will either have to drop or alter the forms we teach to high school students or find some more valid reason for teaching them than that they will be useful to the pupil when he begins the study of physics.

These include, as will be seen from the partial list of findings which follow, simultaneous equations of the first degree in the simple form, the general quadratic equation, complicated cases of factoring and the like.

Findings

1. The word count indicates that certain concepts should be given aditional emphasis. Among these are: *constant, variable, axis, rate, maximum, minimum,* and the series of words connected with *variability* and *proportionality*.

2. College subjects dealing with quantitative data make use of an extensive symbolic language quite different from that used in high school mathematics. Greek letters or initial letters (either capital or small) with primes or subscripts are used in preference to the customary x, y, z, a, b, and c of algebra.

3. The arithmetic computations are much more complicated than the types of computation now advocated for courses in arithmetic.

4. There are comparatively few geometric facts involved in the subjects investigated, and those are the more simple and fundamental facts upon which ample drill is usually given in high school.

5. The ability to make scale drawings including the use of the protractor is essential.

6. The frequency of the four fundamental operations with standard form numbers in expressing large and small numbers justifies the emphasis of this phase of the theory of exponents in high school.

7. There are no complicated polynomials, parentheses, fractional equations, radicals, or complicated cases of factoring found in the texts examined.

8. The general quadratic equation does not occur in Stewart's text. It appears but rarely in the other texts, and when it does occur, it is not in a factorable form.

9. The customary form of simultaneous equations of the first degree does not occur, but the more unusual form with fractions appears frequently in the study of lenses and mirrors.

10. Among the general processes and methods of procedure which are illustrated and discussed, the following are typical:

a. The critical reading of the problem in order to get the exact meaning.

b. Roughly checking the answer to see if it is reasonable.

c. Reduction of data to the proper unit of measure by means of a known ratio before substituting in the appropriate formula.

d. Translations of verbal law or definition into a mathematical formula.

e. Using a verbal rule as a stimulus to a process carried on in imagination.

f. Discarding irrelevant data.

g. Diagramatic representation of relationships between quantities and forces.

12. A study of the results of the test shows the average number of "corrects" on eight hundred fifty-nine papers to be 6.8 out of a possible eighteen. Inability to interpret relationships of letters in formulas, confusion of capital and small letters, confusion of subscripts and exponents, inexperience in solving for a ratio and a reciprocal, failure to roughly check results, and carelessness in leaving a letter in the expression of the value of that letter were among the factors contributing to this low average.

The study is well worth careful reading by any teacher of mathematics or science. A mere list of the chapter headings will give one a definite idea of where many helpful things can be found. I. Summary of Other Investi-

gations Pertaining to this Subject, II. Mathematical Vocabulary, III. Symbolism, IV. Facts, Concepts, and Skills, V. General Processes and Methods of Procedure, VI. Test on Formulas, VII. Summary and Conclusions.

The teacher of mathematics will see that he has not been preparing to any great extent his students for such a course in physics that Professor Congdon had in mind. Professor Congdon was very careful to see to it that the material he used was fairly representative of what is now taught in first year college physics. The physics teacher, who has in so many cases so loudly proclaimed that the students have learned nothing in high school, will see that many things which he expects the children to know about algebra are probably a little beyond the needs and ability of a ninth grade high school pupil. Possibly the physics teacher should take some of the responsibility for teaching children how to handle standard numbers, formulas applying to special types of problems, and the like. This would not excuse the mathematics teacher, of course, from making many changes, as, for example, using other letters than x, y, and z to represent unknown numbers, using subscripts, developing an ability to interpret relationships, encouraging proper habits of checking, and the like.

JAS. H. ZANT
Southeastern State Teachers College
Durant, Oklahoma

Theory of Determinants. By Thomas Muir. Revised and enlarged by Dr. William H. Metzler, Dean of the New York State College for Teachers, Albany, New York. Published privately at six dollars.

This announcement is of particular interest as Dr. Metzler has been for a long time actively associated with the Association of Teachers of Mathematics in the Middle States and Maryland, and was primarily responsible for the success of THE MATHEMATICS TEACHER, acting as its editor-in-chief for many years.

The reviewer knows, by over thirty years' intimate contact with Dr. Metzler's work, that for all of this time he has been preparing the material for this treatise on determinants, —in which department of mathematics he has long been one of the foremost authorities in the world.

At a comparatively early time Dr. Muir, of South Africa, whose work on this subject was then standard, recognized from Dr. Metzler's articles published here and abroad, that he had reached a point in advance of all other investigators in this field, and suggested that he undertake to bring together everything known on the subject with such original contribution as he was then publishing. Dr. Muir at that time authorized Dr. Metzler to use whatever he wished from his own publications, and he has continued to encourage him in this monumental work.

Dr. Metzler has therefore, over this long period, investigated everything extant on determinants, analyzing the books and articles published in various foreign countries as well as in the United States, and has organized and integrated all the varying contributions, filling in gaps and extending investigations whenever his own knowledge showed this to be desirable.

As a result he has published a treatise that will be the authority on this subject for a long time to come. It is a book that no mathematician with any interest in determinants or any pretensions to knowledge concerning this

subject can possibly do without.

The book is not being handled as a commercial proposition, but by Dr. Metzler's generosity, is being sold at less than the cost of production, in order that it may be made available as widely as possible.

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